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Construction of supercharges for the one-dimensional supersymmetric nonlinear sigma model

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Abstract: This paper addresses an issue essential to the study of hidden supersymmetries (meaning here ones that do not close on the Hamiltonian) for one-dimensional non-linear supersymmetric sigma models. The issue relates to ambiguities, due to partial integrations in superspace, both in the actual definition of these supersymmetries and in the Noether definition of the associated supercharges. The unique consistent forms of both these definitions have to be determined simultaneously by a process that adjusts the former definitions so that the associated supercharges do indeed correctly generate them with the aid of the canonical formalism. The paper explains and illustrates these matters and gives some new results.

Keywords: Sigma Models, Extended Supersymmetry, Superspaces.

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1. Introduction

This paper is concerned with the study of certain aspects of one-dimensional supersymmetric sigma models that have not received systematic study elsewhere. It has been well-understood for a long time how superfield methods for such models provide an efficient and clear description of their natural, $N = 1$, supersymmetry. This is generated by a conserved hermitean supercharge Q, which is constructed directly in a problem-free fashion using Noether's theorem and which "closes" on the hamiltonian of the theory via $H = Q^2$. Much the same applies to N-extended supersymmetries when extended superfields are used, the hermitean supercharges $Q_a,\ a = 1, 2, \ldots, N$, closing on the hamiltonian in this context via $\{Q_a, Q_b\} = 2\delta_{ab}H$. In this paper, we study the conditions under which further 'hidden' supersymmetries are present in such models and the consistent determination of the supercharges which generate them.

The main aspects of our work for which novelty and importance is claimed stem in large part from our relaxation of the requirement that all supersymmetries should close (as noted for $N = 1$ and extended supersymmetries) upon the hamiltonian. Rather, when a hidden supersymmetry with generator Q' has been found in a consistent fashion, it is then necessary to compute the constant of the motion K that follows via $K = Q'^2$, i.e. the algebra of hidden supersymmetries has to be determined after the supersymmetries themselves have been discovered. For an example of this situation, one that is so simple as to be free of technical difficulties, see [1]. It is the present widening of the context which gives rise to ambiguities in the path from the action S of the theory to hidden supersymmetries and their generators, causing problems that are of a troublesome sort and of widespread occurrence. We turn next to explaining the nature of the ambiguities and to describing the procedure necessary and sufficient for their unique resolution.

First we consider the contrasting situations for natural and hidden supersymmetries. For the former the transformations in question are known a priori and yield an exactly calculable total divergence for δS, so that Noether's theorem produces (what always turns out to be) the correct supercharge Q without ambiguity. In the latter case one is trying to *determine* hidden supersymmetries, that is to determine conditions on the unknown tensors that feature in some ansatz for them, in such a way that $\delta S = 0$ follows. Some partial integrations in superspace are essential in this process, but wide variations in these, all of which do yield $\delta S = 0$, are possible. They lead to different conditions on the unknown tensors, different total divergence expressions for δS, and hence (via Noether's theorem) to different expressions for the corresponding supercharges. How does one chose the correct procedure and identify the correct results? Clearly a criterion from *outside* the calculation just described is called for and, of course, one exists.

The criterion for the correct resolution of these ambiguities is easily stated. One must perform the integrations in superspace which determine the explicit form of the hidden supersymmetry transformations giving invariances of the action in such a way that the corresponding Noether supercharge generates via the canonical formalism exactly the same supersymmetry transformations. For a natural supersymmetry, where the the transformations are known from the outset, this happens routinely without problem, as already noted. Although it was easy to state our criterion, there is, in the contexts of interest in the present paper, no systematic way to achieve its implementation. Rather, one has to proceed starting from an initial form of the hidden supersymmetry that does give $\delta S = 0$, computing the corresponding Q, observing if (and of course exactly how) it fails to give back the original supersymmetry transformations and adjusting these transformations until one reaches the goal required by the consistency criterion. We emphasise that there is no other way to escape the problems described except in very simple situations. The illustration presented in section 6 gives a good impression of the nature and treatment of the problem in a modestly difficult case. It should be apparent even here that that we are raising an important issue.

We continue our introduction with some background material on sigma models and supersymmetries including extended and hidden ones, in part amplifying the summary just given of the problems on which the paper focuses. A sigma model is an action for dynamical fields considered as maps from a spacetime to a target manifold. In the case where the spacetime is (1+1)-dimensional, the sigma-model can be seen to provide the action for a string world-sheet and to describe the propagation of a string in the target manifold. There is a generalisation to a supersymmetric worldsheet with corresponding target space supersymmetry which describes the propagation of a superstring. Indeed, for these reasons there has been extensive study of such low-dimensional sigma-models. Many interesting features of sigma-models can be illustrated by considering the case in which the spacetime is one-dimensional and parameterised by a real time co-ordinate. This leads to the action for a particle propagating on the target manifold, and the supersymmetric generalisation is clear. It is with such models that we are concerned here.

We shall study the one-dimensional nonlinear sigma-model with $N = 1$ supersymmetry in the case in which the target manifold is a principal fibre bundle $P(\mathcal{M}, G)$. We are particularly interested in the case where G is a compact Lie group. By considering a sigma model involving bosonic $N = 1$ superfields valued in $\mathcal{M}$ and fermionic superfields valued in G, we arrive at an action for a supersymmetric point particle with internal "colour" spin degrees of freedom transforming under G. To complete the picture, we introduce a background Yang-Mills field as the curvature of a connection on P. The colour degrees of freedom are minimally coupled to the bosonic superfields via the gauge potential.

Such a model, by virtue of its superspace construction exhibits an explicit $N = 1$ supersymmetry. There exists a (fermionic) supercharge Q_0 which, upon use of the canonical formalism, generates the canonical $N = 1$ supersymmetry transformations. The explicit nature of this supersymmetry allows the definition of Q_0 via the Noether procedure to proceed without the ambiguities mentioned above. Further use of the classical canonical formalism allows us to calculate the "square" of Q_0 via the Poisson-Dirac bracket, giving the hamiltonian for the theory as $\{Q_0\,,\,Q_0\} = -2iH$ (in the quantum theory there is an extra factor of i on the right hand side). We say that Q_0 "closes" on the hamiltonian.

1.1 Extended supersymmetry

There is by now a very large body of work including [2, 3, 4, 5, 6, 7, 8] on $N = 1$ supersymmetric quantum-mechanical models in which there is a single hermitean supercharge that closes on the hamiltonian. Clearly, great importance is attached to the search for additional supersymmetries in such models. An additional su-

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persymmetry is a set of transformations which leave the action invariant and commute with the original supersymmetry transformations. These will in turn generate an extra (fermionic) supercharge Q', say, which, upon use of the Poisson-Dirac bracket, will then satisfy $\{Q_0\,,\,Q'\}=0$. The bosonic quantity constructed from Q' by $K=\frac{i}{2}\{Q'\,,\,Q'\}$ can be seen, upon use of the Jacobi identity, to be time-invariant.

There is a large body of literature on so-called extended supersymmetry. This is defined as a set of extra supersymmetries of the type described above, all with $K=H$, leading to the algebra

$$\{Q_\alpha, Q_\beta\}=2\delta_{\alpha\beta}H\,, \tag{1.1}$$

in quantum mechanics, or classically, in terms of Poisson brackets,

$$\{Q_\alpha, Q_\beta\}=-2i\delta_{\alpha\beta}H\,. \tag{1.2}$$

There are two possibilities for generating such an extra supersymmetry of the one-dimensional supersymmetric nonlinear sigma-model. The first is related to an endomorphism symmetry of the tangent bundle generated by a complex structure. A complex structure I is a $(1,1)$-tensor on $\mathcal{M}$ which gives a closed two-form on $\mathcal{M}$ by $\omega=gI$, where g is the metric. To be a complex structure, a tensor I must be covariantly constant and satisfy $I^2=-\mathbf{1}$, with $\mathbf{1}$ the identity. A manifold with such a structure is said to be Kähler and must be of real dimension $2n$ for $n\in\mathbb{N}$. This allows the supersymmetry to be extended from $N=1$ to $N=2$. In the case where there are three complex structures I, J and K which satisfy the quaternion algebra, the manifold is hyperKähler and must have real dimension $4n$ for $n\in\mathbb{N}$. The Poisson-Dirac brackets among the set of corresponding supercharges vanish, leaving an $N=4$ extended supersymmetry.

The second type of extended supersymmetry is generated from supersymmetries between the fields on the base manifold $\mathcal{M}$ and fields on the fibre G. This requires $\mathcal{M}$ and G to have the same dimension. Maps between the two can be interpreted as vielbeine. The complex structures (see above) relate the dynamical bosons on $\mathcal{M}$ to the fermions on $\mathcal{M}$ via supersymmetry. The maps between $\mathcal{M}$ and G allow us to relate, via supersymmetry, the dynamical bosons on $\mathcal{M}$ to the fermions on G. Thus these maps play a similar role to that of the complex structures and satisfy similar conditions.

A derivation is given by Coles and Papadopoulos [2] of a complete set of conditions sufficient for invariance of the action under both types of supersymmetry for the one-dimensional supersymmetric nonlinear sigma model.

1.2 Hidden supersymmetries

In this paper, we wish to study the generalisation of extended supersymmetry found by relaxing the condition that the extra supercharges close on the hamiltonian. Use of the Jacobi identity allows us to see that any quantity obtained as a square of such a generalised supercharge commutes with the hamiltonian and is itself a (bosonic) constant of the motion. We describe the conditions under which such generalised "hidden" supersymmetries exist in one-dimensional supersymmetric sigma-models.

Supersymmetry algebras which, as in [1, 9], do not close on H, rather on different important constants of the motion, have been displayed for several theories, including the motion of a spinning particle in a Dirac monopole [1], Kerr-Newman [10] and Taub-NUT [11, 12] background. These involve at least one, one and four additional supercharges respectively. A similar treatment has been applied to the case of a particle with both spin and colour degrees of freedom in a background Yang-Mills field in [13]. The examples cited all involve the use, in an essential way, of Killing-Yano tensors, a topic comprehensively discussed by Tanimoto [14] for a general curved plus electromagnetic background. These are essentially generalisations of the complex structures which are used to generate extended supersymmetry (see above). Due to the fact that the associated supercharges are not required to close on the hamiltonian, they satisfy less restrictive conditions and a much wider class of manifolds admit such supersymmetries. There are also suitable generalisations of the second class of extended supersymmetries described above, as illustrated in [13]. These involve supersymmetries between the dynamical bosons on $\mathcal{M}$ and the dynamical fermions on G and are generalisations of those described in section 1.1.

1.3 Construction of supercharges

In an important paper on the supersymmetries of the one-dimensional supersymmetric nonlinear sigma model, Coles and Papadopoulos [2] provide a list of conditions sufficient for the invariance, $\delta S = 0$, of the action S. We say "sufficient" here because, as is noted in [2], the list contains ambiguities due to the role played by partial integrations in superspace. In relation to invariance, this may well not be of major significance. However, for our purpose, detailed analysis and treatment of such ambiguities is of paramount importance. We require the explicit construction of the supercharge Q associated with each set of supersymmetry transformations $\mathcal{J}$ under which $\delta S = 0$, and we use Noether's theorem to perform it. The ambiguities mentioned above assume real significance at this point since we may well (and in general do) have $\delta S = 0$ and $\delta L \neq 0$, since a total time derivative in δL does not contribute to δS. As the construction of the associated supercharges involves δL, related ambiguities affect the supercharges and manifest themselves when a plausible expression for Q is used within the canonical formalism of the theory to calculate the supersymmetry transformations that Q generates. Consistency requires that the results of these calculations should coincide with the original transformations $\mathcal{J}$ but they may well fail to do so. In all but very simple contexts, it is in general a highly non-trivial matter to handle the partial integrations in exactly the fashion that is required to achieve consistency.

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The paper is organised as follows. In section 2, we describe the supersymmetric formalism of the one-dimensional supersymmetric sigma-model. We describe the canonical quantisation of such a model and the $N = 1$ supersymmetry algebra in section 3. Then, in section 4, we discuss the existence of additional supersymmetries of the model and construct Noether charges for the original and additional supersymmetries. Having exposed the ambiguities involved in calculation of the supercharges, we determine the conditions for invariance of S in the form required for the consistent construction of the supercharges and perform that construction explicitly. This culminates in the display of the conditions for invariance of the action that embody consistent treatment of supercharges, for which explicit expressions are given. Canonical equations in alternative form, useful for some purposes, are mentioned in section 5. In section 6, we describe an example that makes fully explicit the nature and resolution of the problem of consistent calculation of Q.

For simplicity, we present our discussion in the language of classical mechanics; the extension of it to the quantum case proceeds in straightforward fashion.

2. The one-dimensional supersymmetric sigma-model

Sigma-models are theories involving fields considered as maps from spacetime S to a target manifold $\mathcal{M}$. The case $S = \mathbb{R}^{(1,1)}$ has been much-studied as this describes the propagation of a string in the background manifold $\mathcal{M}$. It is also interesting to consider the case of $S = \mathbb{R}$, parameterised by time. This gives a theory of quantum mechanics on the manifold $\mathcal{M}$. If $\mathcal{M}$ is n-dimensional, there exist co-ordinates on $\mathcal{M}$ such that a given field Φ, say, is composed of n maps

$$\Phi^i : S \to \mathcal{M} \,. \tag{2.1}$$

This will be taken to be an $N = 1$ superfield with bosonic components x^i and fermionic components ψ^i. We introduce co-ordinates (t, θ) on the one-dimensional, $N = 1$ superspace, with t real and θ a Grassmann parameter and write

$$\Phi^i = x^i(t) + i\theta\psi^i(t) \,. \tag{2.2}$$

The model studied in this paper is that where the target space is a principal bundle $P(\mathcal{M}, G)$. Dynamical fermions, valued in the fibre, are defined by the fermionic superfield $\Lambda^\alpha = \lambda^\alpha + \theta F^\alpha$, where λ^α are fermionic and the F^α are auxiliary bosonic fields. We use the superderivative $D = \partial_\theta - i\theta\partial_t$ to write down an action for the fields Φ^i and Λ^α, with minimal coupling:

$$S = \int d\theta \, dt \mathcal{L} \,, \tag{2.3}$$

$$\mathcal{L} = \frac{1}{2}\left(i g_{ij}(\Phi)\dot{\Phi}^i D\Phi^j - \frac{1}{3} c_{ijk}(\Phi) D\Phi^i D\Phi^j D\Phi^k + h_{\alpha\beta}\Lambda^\alpha \nabla \Lambda^\beta \right) . \tag{2.4}$$

This involves the covariant derivative $\nabla\Lambda^\alpha = D\Lambda^\alpha + A_i{}^\alpha{}_\beta(\Phi)D\Phi^i\Lambda^\beta$, where $A_i{}^\alpha{}_\beta$ is a gauge connection with field strength

$$F_{ij}{}^\alpha{}_\beta = \partial_i A_j{}^\alpha{}_\beta - \partial_j A_i{}^\alpha{}_\beta + A_i{}^\alpha{}_\gamma A_j{}^\gamma{}_\beta - A_j{}^\alpha{}_\gamma A_i{}^\gamma{}_\beta \,. \tag{2.5}$$

The second term of (2.4) arises from a partial integration in superspace of the Wess-Zumino term $\frac{i}{2}b_{ij}\dot{\Phi}^i D\Phi^j$, so that the c_{ijk} are the components of the 3-form $c = -\frac{3}{2}db$. c can be interpreted as the torsion of the manifold $\mathcal{M}$. The fibre is a compact Lie group and hence the metric on the fibre can be taken to be $h_{\alpha\beta} = \delta_{\alpha\beta}$. Writing $\mathcal{L} = K + \theta L$, we have

$$K = \frac{1}{2}\left(-g_{ij}\dot{x}^i\psi^j + \frac{i}{3}c_{ijk}\psi^i\psi^j\psi^k + h_{\alpha\beta}\lambda^\alpha F^\beta - ih_{\alpha\beta}\lambda^\alpha A_i{}^\beta{}_\gamma\lambda^\gamma\psi^i\right), \tag{2.6}$$

$$\begin{aligned} L = \frac{1}{2}\Big(& g_{ij}\dot{x}^i\dot{x}^j + ig_{ij}\psi^i\dot{\psi}^j - ig_{ij,k}\psi^k\dot{x}^i\psi^j - ic_{ijk}\dot{x}^i\psi^j\psi^k - \\ & - \frac{1}{3}c_{ijk,n}\psi^n\psi^i\psi^j\psi^k + ih_{\alpha\beta}\lambda^\alpha\dot{\lambda}^\beta + h_{\alpha\beta}F^\alpha F^\beta + ih_{\alpha\beta,i}\psi^i\lambda^\alpha F^\beta - \\ & - ih_{\alpha\beta}A_i{}^\beta{}_\gamma\left(F^\alpha\lambda^\gamma - \lambda^\alpha F^\gamma\right)\psi^i + h_{\alpha\beta}A_i{}^\beta{}_{\gamma,k}\psi^k\lambda^\alpha\lambda^\gamma\psi^i + \\ & + ih_{\alpha\beta}A_i{}^\beta{}_\gamma\lambda^\alpha\lambda^\gamma\dot{x}^i + h_{\alpha\beta,j}\psi^j A_i{}^\beta{}_\gamma\lambda^\alpha\lambda^\gamma\psi^i\Big) \,. \end{aligned} \tag{2.7}$$

The fields F^α are non-dynamical and can be eliminated using their Euler-Lagrange equations. Using this, the condition $\nabla h = 0$ and the fact that c is a closed 3-form, $dc = 0$, we have

$$\begin{aligned} L = {} & \frac{1}{2}\left(g_{ij}\dot{x}^i\dot{x}^j + ig_{ij}\psi^i\dot{\psi}^j + ig_{ij,k}\psi^j\psi^k\dot{x}^i - ic_{ijk}\dot{x}^i\psi^j\psi^k + ih_{\alpha\beta}\lambda^\alpha\dot{\lambda}^\beta\right) + \\ & + \frac{i}{2}A_{i\alpha\beta}\lambda^\alpha\lambda^\beta\dot{x}^i + \frac{1}{4}F_{ij\alpha\beta}\lambda^\alpha\lambda^\beta\psi^i\psi^j \,. \end{aligned} \tag{2.8}$$

The lagrangean (2.4) is written in terms of $N = 1$ superfields and, as such, has an explicit $N = 1$ supersymmetry given in terms of a Grassmann parameter ϵ by

$$\delta t = -i\epsilon\theta \,, \qquad \delta\theta = -\epsilon \,. \tag{2.9}$$

The generator of these transformations is the supercharge Q_0. This is fermionic and has an explicit expression in terms of the dynamical fields of the model. We can give the canonical formalism for the lagrangean in the usual way via the Poisson-Dirac bracket and show that the supercharge generates the correct transformations of the dynamical variables via the Poisson-Dirac bracket. This will be described in detail below. The square of the supercharge, computed via the Poisson-Dirac bracket, gives the classical hamiltonian by

$$\{Q_0 \,, Q_0\} = -2iH \,. \tag{2.10}$$

The analogue of this statement in quantum mechanics is with the right-hand side multiplied by i. It should be noted that this procedure does generate the correct hamiltonian; this can be seen by calculating it in the conventional way as

$$H = \sum \dot{X} \frac{\partial L}{\partial \dot{X}} - L \,, \tag{2.11}$$

where the sum is over all dynamical variables. The time evolution of an arbitrary quantity K is then given by

$$\frac{dK}{dt} = \{K \,,\, H\} \,. \tag{2.12}$$

This leads to the important observation that any quantity Q' which satisfies $\{Q_0 \,,\, Q'\} = 0$ generates a constant of the motion via $K = -\frac{i}{2}\{Q' \,,\, Q'\}$ because $\{K \,,\, H\} = 0$ upon use of the Jacobi identity.

3. Canonical quantisation

The quantisation of this model follows familiar lines. From (2.8) we derive the following conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\dot{x}^j + \frac{i}{2} g_{ij,k}\psi^j\psi^k + \frac{i}{2} A_{i\alpha\beta}\lambda^\alpha\lambda^\beta - \frac{i}{2} c_{ijk}\psi^j\psi^k \,, \tag{3.1}$$

$$\tau_i = \frac{\partial L}{\partial \dot{\psi}^i} = -\frac{i}{2} g_{ij}\psi^j \,, \tag{3.2}$$

$$\xi_\alpha = \frac{\partial L}{\partial \dot{\lambda}^\alpha} = -\frac{i}{2} h_{\alpha\beta}\lambda^\beta \,. \tag{3.3}$$

Thus we have two constraint functions

$$\eta_i = \tau_i + \frac{i}{2} g_{ij}\psi^j \,, \tag{3.4}$$

$$\sigma_\alpha = \xi_\alpha + \frac{i}{2} h_{\alpha\beta}\lambda^\beta \,. \tag{3.5}$$

We use the fundamental brackets

$$\{x^i, p_j\} = \delta^i{}_j \,,\; \{\psi^i, \tau_j\} = -\delta^i{}_j \,,\; \{\lambda^\alpha, \xi_\beta\} = -\delta^\alpha{}_\beta \,, \tag{3.6}$$

to obtain

$$\{\eta_i, \eta_j\} = -i g_{ij} \,, \tag{3.7}$$

$$\{\sigma_\alpha, \sigma_\beta\} = -i h_{\alpha\beta} \,, \tag{3.8}$$

and define the Dirac bracket $\{A, B\}^*$ by

$$\{A, B\}^* = \{A, B\} - \{A, \eta_i\} i g^{ij} \{\eta_j, B\} - \{A, \sigma_\alpha\} i h^{\alpha\beta} \{\sigma_\beta, B\} \,. \tag{3.9}$$

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Since, from now on, all brackets will be Dirac brackets, the asterisk is left implicit. We also work with the covariant momentum

$$\pi_i = g_{ij}\dot{x}^j = p_i - \frac{i}{2}g_{ij,k}\psi^j\psi^k - \frac{i}{2}A_{i\alpha\beta}\lambda^\alpha\lambda^\beta + \frac{i}{2}c_{ijk}\psi^j\psi^k \,. \tag{3.10}$$

We then have the following canonical equations

$$\begin{aligned}
&\{x^i, x^j\} = 0 && \{x^i, \pi_j\} = \delta^i_j && \{x^i, \psi^j\} = 0\,,\\
&\{x^i, \lambda^\alpha\} = 0 && \{\psi^i, \pi_j\} = -\Gamma^i{}_{jk}\psi^k - c^i{}_{jk}\psi^k && \{\lambda^\alpha, \pi_i\} = -A_i{}^\alpha{}_\beta\lambda^\beta\,,\\
&\{\psi^i, \psi^j\} = -ig^{ij} && \{\lambda^\alpha, \lambda^\beta\} = -ih^{\alpha\beta} && \{\psi^i, \lambda^\alpha\} = 0\,,\\
&\{\pi_i, \pi_j\} = \frac{i}{2}R_{ijpq}\psi^p\psi^q + \frac{i}{2}F_{ij\alpha\beta}\lambda^\alpha\lambda^\beta + i\nabla_{[j}c_{i]pq}\psi^p\psi^q - ic_{inp}c_j{}^n{}_q\psi^p\psi^q\,,
\end{aligned} \tag{3.11}$$

where $F_{ij\alpha\beta}$ is defined in (2.5) and

$$R_{ijpq} = g_{in}\left(\partial_p\Gamma^n{}_{jq} - \partial_q\Gamma^n{}_{jp} + \Gamma^n{}_{kp}\Gamma^k{}_{jq} - \Gamma^n{}_{kq}\Gamma^k{}_{jp}\right). \tag{3.12}$$

We define a generalised connection $\tilde{\Gamma}^i_{jk} = \Gamma^i{}_{jk} + c^i{}_{jk}$ and a corresponding generalised curvature $\tilde{R}_{ijpq}$, defined from $\tilde{\Gamma}$ as R is from Γ, to obtain

$$\tilde{R}_{ijpq} = R_{ijpq} + 2\nabla_{[j}c_{i]pq} - 2c_{inp}c_j{}^n{}_q\,, \tag{3.13}$$

so that the brackets (3.11) lead to the form for the general Dirac bracket

$$\begin{aligned}
\{A, B\} = {} & \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial \pi_i} - \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial x^i} + \\
& + \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial \pi_j}\left(\frac{i}{2}\tilde{R}_{ijpq}\psi^p\psi^q + \frac{i}{2}F_{ij\alpha\beta}\lambda^\alpha\lambda^\beta\right) - \\
& - \left((-)^b\frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial \psi^j} - (-)^{a+b}\frac{\partial A}{\partial \psi^j}\frac{\partial B}{\partial \pi_i}\right)\tilde{\Gamma}^j_{ik}\psi^k - \\
& - \left((-)^b\frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial \lambda^\alpha} - (-)^{a+b}\frac{\partial A}{\partial \lambda^\alpha}\frac{\partial B}{\partial \pi_i}\right)A_i{}^\alpha{}_\beta\lambda^\beta + \\
& + i(-)^a\frac{\partial A}{\partial \psi^i}\frac{\partial B}{\partial \psi^j}g^{ij} + i(-)^a\frac{\partial A}{\partial \lambda^\alpha}\frac{\partial B}{\partial \lambda^\beta}h^{\alpha\beta}\,,
\end{aligned} \tag{3.14}$$

where a and b are the Grassmann parities of A and B respectively.

4. Construction of supercharges

4.1 The construction of Noether charges

We construct supercharges from general superfield transformations $\delta\Phi^i$ and $\delta\Lambda^\alpha$ which leave the superfield action (2.3) invariant. By Noether's theorem, there must exist a conserved supercharge which generates each such set of transformations. Below is a description of the construction of these supercharges. An explicit example

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is given in section 6, which is intended to illustrate the principles involved. We set out from the action (2.3)

$$S = \int d\theta\, dt \mathcal{L} = \int dt L\,,$$
$$\mathcal{L} \equiv \mathcal{L}(\Phi, D\Phi, \dot{\Phi}, \Lambda, D\Lambda)\,.$$

We now calculate δL in two ways. First, we apply the transformations of Φ and Λ directly to the action and integrate over the Grassmann variable θ to get δL in the form

$$\delta L = \frac{dJ}{dt} + \{\text{other terms}\}\,. \tag{4.1}$$

The "other terms" are required to vanish, leaving just the total derivative. This requirement imposes a set of conditions on the initial superfield transformations, which we examine in section 4.3. This leaves δL as a total derivative which ensures that the action is invariant. We can also get an expression for δL within the standard procedure for Noether's theorem in the form

$$\delta L = \partial_t \left(\sum_X \delta X \frac{\partial L}{\partial \dot{X}} \right), \tag{4.2}$$

where the sum is over the dynamical variables x, ψ and λ. Equating (4.1) and (4.2), we have a time-invariant supercharge Q given by

$$i\epsilon Q = \sum_X \delta X \frac{\partial L}{\partial \dot{X}} - J\,, \tag{4.3}$$
$$\partial_t Q = 0\,. \tag{4.4}$$

In the above expression for Q, ϵ is the constant, Grassmann-odd parameter which appears in the supersymmetry transformations. As indicated in the Introduction, difficulties in implementing the procedure outlined arise in supersymmetric theories from the possibility of integrating by parts in superspace, so that the separation of (4.1) into the two indicated pieces is not unique. Given a version of (4.1), a further such integration may change both J and the "other terms" and hence the conditions under which the latter vanish.

4.2 The fundamental supercharge and the hamiltonian

The lagrangean (2.4) is of the form $\mathcal{L} \equiv \mathcal{L}(\Phi, D\Phi, \dot{\Phi}, \Lambda, D\Lambda)$. Consequently, it is manifestly invariant under the original supersymmetry transformations of the theory (2.9), which are realised on the superfields as

$$\delta\Phi^i = -\epsilon\mathcal{D}\Phi^i\,, \tag{4.5}$$
$$\delta\Lambda^\alpha = -\epsilon\mathcal{D}\Lambda_\alpha\,, \tag{4.6}$$

where ϵ is a hermitean Grassmann variable and $\mathcal{D} = \partial_\theta + i\theta\partial_t$. As $\{D, \mathcal{D}\} = 0$, we also have

$$\delta(D\Phi^i) = -\epsilon\mathcal{D}(D\Phi^i)\,, \tag{4.7}$$
$$\delta(D\Lambda^\alpha) = -\epsilon\mathcal{D}(D\Lambda^\alpha)\,. \tag{4.8}$$

Using the procedure of section 4.1, we can construct the fundamental supercharge Q_0 from these transformations. This yields

$$Q_0 = \pi_i\psi^i - \frac{i}{3}c_{ijk}\psi^i\psi^j\psi^k\,, \tag{4.9}$$

without any ambiguity arising. The supercharge is generated by transformations of the dynamical variables x, ψ and λ so we can check the form of Q_0 by calculating the transformations of Φ^i and λ^α which it generates,

$$i\epsilon\{Q_0, \Phi^i\} = -\epsilon\mathcal{D}\Phi^i = \delta\Phi^i\,, \tag{4.10}$$
$$i\epsilon\{Q_0, \lambda^\alpha\} = -\epsilon F^\alpha = \delta\lambda^\alpha\,. \tag{4.11}$$

Further, we easily perform a canonical calculation of the hamiltonian

$$\{Q_0, Q_0\} = -2iH\,, \tag{4.12}$$

and we find that

$$H = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j - \frac{1}{4}F_{ij\alpha\beta}\psi^i\psi^j\lambda^\alpha\lambda^\beta\,. \tag{4.13}$$

This reproduces, as expected, the canonical result

$$H = \sum \dot{X}\frac{\partial L}{\partial \dot{X}} - L\,. \tag{4.14}$$

4.3 Hidden supersymmetries

To construct further supercharges, we must consider other superfield transformations which leave the action invariant. Following Coles and Papadopoulos [2], the most general such transformations are

$$\delta\Phi^i = \epsilon I^i{}_j D\Phi^j + i\epsilon e^i{}_\alpha\Lambda^\alpha\,, + \tag{4.15}$$
$$\begin{aligned}\delta\Lambda^\alpha &= \epsilon I^\alpha{}_\beta\nabla\Lambda^\beta - A_i{}^\alpha{}_\beta\delta\Phi^i\Lambda^\beta - \epsilon e_i{}^\alpha\dot{\Phi}^i \\ &\quad + i\epsilon E^\alpha{}_{ij}D\Phi^i D\Phi^j + i\epsilon M^\alpha{}_{\beta\gamma}\Lambda^\beta\Lambda^\gamma + i\epsilon G^\alpha{}_{\beta i}\Lambda^\beta D\Phi^i\,.\end{aligned} \tag{4.16}$$

Here, $I^i{}_j$ and $I^\alpha{}_\beta$ are endomorphisms of the sigma model manifold $\mathcal{M}$ and the fibre G respectively; $e^i{}_\alpha$ and $e_i{}^\alpha$ are bundle maps between the manifold and the fibre. In the case where $\dim\mathcal{M} = \dim G$ and $h_{\alpha\beta}$ is the flat metric on G, the e can be interpreted as vielbeine. It should be noted that terms in $I^\alpha{}_\beta$ do not appear in $\delta\lambda_\alpha$. Thus the $I^\alpha{}_\beta$ do not affect the dynamical variables and consequently do not appear in the supercharges. This explains the absence of terms involving $I^\alpha{}_\beta$ from the conditions below. As described in section 4.1, we can obtain a set of conditions, such as those presented in [2], *sufficient* for the invariance of the action under (4.15) and (4.16), these being determined *only up to partial integrations in superspace*. Demanding that the supercharges we construct do generate the original supersymmetry transformations requires the set of conditions on the fields appearing in (4.15) and (4.16) to be in *exactly* the following form:

- Conditions associated with I

$$I_{(ij)} = 0\,, \tag{4.17}$$
$$\nabla_i I_{jk} + \nabla_{[k} I_{j]i} + \frac{3}{2} c_{mjk} I^m{}_i = 0\,, \tag{4.18}$$
$$I^m{}_{[i} F_{j]m\alpha\beta} = 0\,, \tag{4.19}$$
$$G_{\alpha\beta i} = 0\,. \tag{4.20}$$

- Conditions associated with e

$$h_{\alpha\beta} e_i{}^\beta - g_{ij} e^j{}_\alpha = 0\,, \tag{4.21}$$
$$M_{\alpha\beta\gamma} - M_{[\alpha\beta\gamma]} = 0\,, \tag{4.22}$$
$$e^k{}_\alpha c_{kij} + \nabla_i e_{j\alpha} + 2E_{\alpha ij} = 0\,, \tag{4.23}$$
$$\nabla_{[i}\left(c_{jk]m} e^m{}_\alpha\right) + \frac{1}{2} e_{[i}{}^\beta F_{jk]\alpha\beta} = 0\,, \tag{4.24}$$
$$\frac{2}{3}\nabla_i M_{\alpha\beta\gamma} + F_{ij[\alpha\beta} e^j{}_{\gamma]} = 0\,. \tag{4.25}$$

In particular, we note the results, from (4.18) and (4.23) respectively, that

$$c_{n[ij} I^n{}_{k]} = 0\,, \tag{4.26}$$
$$\nabla_{(i} e_{j)\alpha} = 0\,. \tag{4.27}$$

To construct the supercharges, we proceed as above. However, the transformations decouple into the parts generated by the endomorphisms I and the parts generated by the bundle maps e (and the three-form M). These can therefore be treated independently in the construction of supercharges. We will call these supersymmetries type I and type II respectively. We will use the notation $Q_1(I)$ for a type I supersymmetry generated by I and $Q_2(e, M)$ for a type II supersymmetry generated by e and M. Thus, to construct $Q_1(I)$, we use

$$\delta\Phi^i = \epsilon I^i{}_j D\Phi^j\,,$$
$$\delta\Lambda^\alpha = \epsilon I^\alpha{}_\beta \nabla\Lambda^\beta - A_i{}^\alpha{}_\beta \delta\Phi^i \Lambda^\beta\,,$$

which give the type I supercharge

$$Q_1(I) = \pi_i I^i{}_j \psi^j - \frac{i}{3}\left(\nabla_i I_{jk}\right)\psi^i\psi^j\psi^k\,. \tag{4.28}$$

To construct $Q_2(e,M)$, we use

$$\begin{aligned}
\delta\Phi^i &= i\epsilon e^i{}_\alpha \Lambda^\alpha\,,\\
\delta\Lambda^\alpha &= -A_i{}^\alpha{}_\beta \delta\Phi^i\Lambda^\beta - \epsilon e_i{}^\alpha \dot{\Phi}^i + i\epsilon E^\alpha{}_{ij} D\Phi^i D\Phi^j + i\epsilon M^\alpha{}_{\beta\gamma}\Lambda^\beta\Lambda^\gamma\,,
\end{aligned}$$

which give the type II supercharge

$$Q_2(e,M) = \pi_i e^i{}_\alpha \lambda^\alpha + iE_{\alpha ij}\lambda^\alpha\psi^i\psi^j - \frac{i}{3}M_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma\,. \tag{4.29}$$

At this point, we can verify the correctness of our result by showing that the canonical brackets of Q_1 and Q_2 with the dynamical variables do indeed generate the required transformations of these variables, as in (4.10) and (4.11). Non-trivial calculations allow the following results to be verified

$$\{Q_0, Q_1(I)\} = 0\,, \tag{4.30}$$

$$\{Q_0, Q_2(e,M)\} = 0\,. \tag{4.31}$$

Of course, it should be emphasised that, in presenting the results (4.17) to (4.25), we have already fixed the detail in them so that the consistency arguments just described work correctly. Results (4.28) and (4.29) are new here. If one specialises the results of [2] to our (somewhat less general) context, we see that (4.18) and (4.24) contain important refinements of these results. These refinements are critical to the problem of determining the supercharges associated to the extra supersymmetries which we describe.

5. A simplification

If we are prepared to break manifest covariance in the expressions for the supercharges then it is possible to simplify some subsequent calculations significantly. To this end we define $\tilde{\pi}$, by

$$\tilde{\pi}_i = \pi_i - \{c \text{ terms }\}\,, \tag{5.1}$$

$$\tilde{\pi}_i = p_i - \frac{i}{2}g_{ij,k}\psi^j\psi^k - \frac{i}{2}A_{i\alpha\beta}\lambda^\alpha\lambda^\beta\,, \tag{5.2}$$

so that, upon use of (4.23) and (4.26), we obtain

$$Q_0 = \tilde{\pi}_i\psi^i + \frac{i}{6}c_{ijk}\psi^i\psi^j\psi^k\,, \tag{5.3}$$

$$Q_1 = \tilde{\pi}_i I^i{}_j\psi^j - \frac{i}{3}\left(\nabla_i I_{jk}\right)\psi^i\psi^j\psi^k\,, \tag{5.4}$$

$$Q_2 = \tilde{\pi}_i e^i{}_\alpha\lambda^\alpha - \frac{i}{2}\left(\nabla_i e_{j\alpha}\right)\psi^i\psi^j\lambda^\alpha - \frac{i}{3}M_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma\,. \tag{5.5}$$

In terms of x, $\tilde{\pi}$, ψ and λ, the general Dirac bracket simplifies to

$$\begin{aligned}
\{A, B\} = & \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial \tilde{\pi}_i} - \frac{\partial A}{\partial \tilde{\pi}_i}\frac{\partial B}{\partial x^i} + \\
& + \frac{\partial A}{\partial \tilde{\pi}_i}\frac{\partial B}{\partial \tilde{\pi}_j}\left(\frac{i}{2}R_{ijpq}\psi^p\psi^q + \frac{i}{2}F_{ij\alpha\beta}\lambda_\alpha\lambda_\beta\right) - \\
& - \left((-)^b\frac{\partial A}{\partial \tilde{\pi}_i}\frac{\partial B}{\partial \psi^j} - (-)^{a+b}\frac{\partial A}{\partial \psi^j}\frac{\partial B}{\partial \tilde{\pi}_i}\right)\Gamma^j{}_{ik}\psi^k - \\
& - \left((-)^b\frac{\partial A}{\partial \tilde{\pi}_i}\frac{\partial B}{\partial \lambda_\alpha} - (-)^{a+b}\frac{\partial A}{\partial \lambda_\alpha}\frac{\partial B}{\partial \tilde{\pi}_i}\right)A_{i\alpha\beta}\lambda_\beta + \\
& + i(-)^a\frac{\partial A}{\partial \psi^i}\frac{\partial B}{\partial \psi^j}g^{ij} + i(-)^a\frac{\partial A}{\partial \lambda_\alpha}\frac{\partial B}{\partial \lambda_\beta}h^{\alpha\beta}\,.
\end{aligned} \tag{5.6}$$

It is helpful and appropriate to employ (5.4) and (5.5) in the discussion of the supercharge algebra via the classical Poisson-Dirac bracket. We then have the conditions:

- Commutation of two type I supercharges; $\{Q_1(I)\,,\, Q_1'(J)\} = 0$:

$$I_{m(i}J_{j)}{}^m = 0\,, \tag{5.7}$$

$$I^m{}_{[i}J^n{}_{j]}F_{mn\alpha\beta} = 0\,, \tag{5.8}$$

$$\mathcal{N}(I,J)^i{}_{jk} \equiv \left(I_{m[j}\nabla^m J^i{}_{k]} - I^i{}_m\nabla_{[j}J^m{}_{k]}\right) + (I \leftrightarrow J) = 0\,, \tag{5.9}$$

$$R_{mn[ij}I^m{}_kJ^n{}_{l]} = 0\,, \tag{5.10}$$

$$J_{m[i}\nabla^m\gamma^{(I)}_{jkl]} + \frac{3}{2}\gamma^{(I)}_{m[ij}\gamma^{(J)m}_{kl]} + (I \leftrightarrow J, \gamma^{(I)} \leftrightarrow \gamma^{(J)}) = 0\,, \tag{5.11}$$

  where $\gamma^{(I)}_{ijk} = \nabla_{[i}I_{jk]}$. The $(1,2)$-tensor $\mathcal{N}^i{}_{jk}$ is the Nijenhuis concomitant of the two endomorphisms I and J.

- Commutation of two type II supercharges; $\{Q_2(e,M)\,,\, Q_2'(f,N)\} = 0$:

$$e^{(i}{}_\alpha f^{j)\alpha} = 0\,, \tag{5.12}$$

$$M_{\mu[\alpha\beta}N_{\gamma\sigma]}{}^\mu - \frac{1}{2}F_{ij[\alpha\beta}e^i{}_\gamma e^j{}_{\sigma]} = 0\,, \tag{5.13}$$

$$f_{n[\alpha}\nabla^n e^i{}_{\beta]} + f^{i\mu}M_{\mu\alpha\beta} + (e \leftrightarrow f, M \leftrightarrow N) = 0\,, \tag{5.14}$$

$$e_{[i}{}^\alpha\nabla_j f_{k]\alpha} + f_{[i}{}^\alpha\nabla_j e_{k]\alpha} = 0\,, \tag{5.15}$$

$$e_{[i}{}^\alpha f_j{}^\beta F_{kl]\alpha\beta} = 0\,, \tag{5.16}$$

$$f_{[i}{}^\gamma F_{j]k\alpha[\beta}e^k{}_{\gamma]} + (e \leftrightarrow f) = 0\,. \tag{5.17}$$

- Commutation of type I and type II supercharges; $\{Q_1(I)\,,\, Q_2(e,M)\} = 0$:

$$e^k{}_\alpha\nabla_k I_{ij} + 2I_{k[i}\nabla^k e_{j]\alpha} = 0\,, \tag{5.18}$$

$$I^n{}_{[i}F_{jk]\alpha\beta}e_{n\beta} = 0\,. \tag{5.19}$$

We look for the maximal commuting set $\{Q_{1i}\,,\, Q_{2j}\}$ as the maximal extension of the supersymmetry algebra.

6. An example

We describe here the treatment of an explicit example in order to illustrate the subtleties associated with the construction of the supercharges that are the central focus of this paper. Consider the special case of (2.4)

$$S_0 = \int d\theta\, dt \frac{1}{2} i g_{ij}(\Phi)\dot{\Phi}^i D\Phi^j\,, \tag{6.1}$$

and (4.15)

$$\delta\Phi^i = \epsilon I^i{}_j(\Phi) D\Phi^j\,. \tag{6.2}$$

We compute δS_0 directly, finding five terms. We first treat the two terms which do not involve any derivatives of g_{ij} or I_{ij}. Using an integration by parts to derive the second line, we find

$$\delta S_0 = \frac{1}{2} i \int d\theta\, dt \epsilon \left[\cdots I_{ij}(\Phi) D\dot{\Phi}^j D\Phi^i - I_{ij}(\Phi)\dot{\Phi}^i D^2\Phi^j + \cdots\right] \tag{6.3}$$

$$= \frac{1}{2} i \int d\theta\, dt \epsilon \left[\cdots D\left(I_{ij}(\Phi) D\Phi^i \dot{\Phi}^j\right) - I_{ij,k}(\Phi) D\Phi^k D\Phi^i \dot{\Phi}^j\right]\,. \tag{6.4}$$

To reach this point, we have eliminated terms of the form $I_{ij}\dot{\Phi}^i\dot{\Phi}^j$ by imposing the condition

$$I_{ij} + I_{ji} = 0\,, \tag{6.5}$$

where $I_{ij} = g_{ik} I^k{}_j$. The divergence term in (6.4) does not contribute to δS_0 but does contribute to δL_0 and hence, via Noether's theorem, to the supercharge $\tilde{Q}$ associated with (6.2). We next collect the remaining terms so as to absorb derivatives of g_{ij} into Christoffel symbols, obtaining

$$\delta S_0 = -\frac{1}{2}\epsilon \int d\theta\, dt \left(I_{jk,i} - 2\Gamma^p{}_{ij} I_{pk}\right) D\Phi^j D\Phi^k D^2\Phi^i\,. \tag{6.6}$$

Demanding that the part of the bracket in (6.6) antisymmetric in j and k vanishes is sufficient to ensure that δS_0 vanishes. However, a direct calculation of the supercharge $\tilde{Q}$ using Noether's theorem leads to a form of $\tilde{Q}$ that fails to reproduce the original transformation (6.2) canonically. The nature of the failure prompts us to split the first term of (6.6) using the identity

$$I_{jk,i} D\Phi^j D\Phi^k D^2\Phi^i = \frac{1}{3} D\left[I_{jk,i} D\Phi^j D\Phi^k D\Phi^i\right] + \frac{2}{3}\left(I_{jk,i} + I_{ji,k}\right) D\Phi^j D\Phi^k D^2\Phi^i\,, \tag{6.7}$$

bringing in a total derivative term of the type that is needed to improve (and, it turns out, to correct) the Noether expression for $\tilde{Q}$. Again, the total derivative term (which is merely the result of an integration by parts) does not contribute to δS_0 but does contribute to $\tilde{Q}$. It now follows that δS_0, including the connection term from (6.6), is given by

$$-\frac{1}{3}\epsilon \int d\theta\, dt \left[\nabla_i I_{jk} + \frac{1}{2}\nabla_k I_{ji} - \frac{1}{2}\nabla_j I_{ki}\right] D\Phi^k D\Phi^j D^2\Phi^i\,, \tag{6.8}$$

where ∇ is the metric-covariant derivative. Thus $\delta S_0 = 0$ can be realised by imposing the condition

$$\nabla_i I_{jk} + \nabla_{[k} I_{j]i} = 0\,. \tag{6.9}$$

This result, contained in the condition (4.18) arising in the general case above, represents a crucial modification of the corresponding equation of [2].

The supercharge $\tilde{Q}$ as defined in (4.3) is

$$i\epsilon\tilde{Q} = \sum_X \delta X \frac{\partial L}{\partial \dot{X}} - J\,, \tag{6.10}$$

where the sum is over all dynamical variables and J is as calculated above, that is

$$J = \epsilon \int d\theta\, dt \left[-\frac{i}{2} D \left(I_{ij}(\Phi) D\Phi^i \dot{\Phi}^j \right) - \frac{1}{6} D \left(\nabla_i I_{jk}(\Phi) D\Phi^i D\Phi^j D\Phi^k \right) \right], \tag{6.11}$$

$$= \frac{i\epsilon}{2} I_{ij}(x) \dot{x}^i \psi^j + \frac{\epsilon}{6} \nabla_i I_{jk}(x) \psi^i \psi^j \psi^k\,. \tag{6.12}$$

Explicit calculation of the expression (6.10) yields the following expression for the supercharge $\tilde{Q}$,

$$\tilde{Q} = I_{ij}(x) \dot{x}^i \psi^j - \frac{i}{3} \nabla_i I_{jk}(x) \psi^i \psi^j \psi^k\,. \tag{6.13}$$

in agreement with that presented above in (4.28). Finally, use of the canonical formalism of the theory (see section 3) allows us to verify the central result that, with precisely the condition (6.9) on I_{ij}, the supercharge $\tilde{Q}$ does indeed generate the original supersymmetry transformation (6.2), that is

$$-i\epsilon\{Q, \Phi^i\} = \epsilon I^i{}_j D\Phi^j = \delta\Phi^i\,. \tag{6.14}$$

Of course, one does not know that the job of determining the exact conditions that must be imposed upon $I^i{}_j$ is indeed complete until a Noether charge has been computed and seen to satisfy (6.14). The results of (4.17) to (4.25) were in fact obtained by generalising the procedure followed in this section, being so arranged as yield Noether charges which generate (4.15) and (4.16) canonically.

7. Conclusion

We have considered the $N = 1$ supersymmetric nonlinear sigma-model and described the conditions under which extra supersymmetries of the most general type can exist. We have derived the conditions for invariance of the action, which were defined up to partial integrations in superspace and shown that there is a unique form of these which is required for the construction, by Noether's theorem, of the supercharges. The precise form of these is determined by imposing the necessary requirement that the supercharges generate the original supersymmetry transformations. We explicitly constructed supercharges for this model and investigated their algebra via the canonical Poisson-Dirac bracket of the theory.

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